Regular variation and differential equations: Theory and methods

Pavel Řehák
Institute of Mathematics
Czech Academy of Sciences

External meeting IM CAS, Brno, November 2016
Structure of the talk

- Regular variation
  - Karamata theory
  - De Haan theory

- Regular variation and differential equations

- Selected results
  - Emden-Fowler type systems
  - Half-linear differential equations
Theory of regularly varying functions

- initiated by J. Karamata (1930). But there are also earlier works...
- study of relations such that

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0, \infty), \quad \forall \lambda > 0,$$

together with their applications (integral transforms – Tauberian theorems, probability theory, number theory, complex analysis, differential equations, etc.)


Definition

A measurable function \( f : [a, \infty) \rightarrow (0, \infty) \) is called regularly varying (at \( \infty \)) of index \( \vartheta \) if for all \( \lambda > 0 \),

\[
\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta}.
\]

[Notation: \( f \in RV(\vartheta) \)]

If \( \vartheta = 0 \), then \( f \) is called slowly varying.

[Notation: \( f \in SV \)]

[\( RV_0 \) means regular variation at zero.]

The Uniform Convergence Theorem

If \( f \in RV(\vartheta) \), then the relation in the definition holds uniformly on each compact \( \lambda \)-set in \((0, \infty)\).
Representation Theorem

- \( f \) is regularly varying of index \( \vartheta \) if and only if
  \[
  f(t) = \varphi(t) \exp \left\{ \int_a^t \frac{\delta(s)}{s} \, ds \right\}
  \]
  where \( \varphi(t) \to \text{const} > 0 \) and \( \delta(t) \to \vartheta \) as \( t \to \infty \).

- \( f \) is regularly varying of index \( \vartheta \) if and only if
  \[
  f(t) = t^\vartheta \varphi(t) \exp \left\{ \int_a^t \frac{\psi(s)}{s} \, ds \right\}
  \]
  where \( \varphi(t) \to \text{const} > 0 \) and \( \psi(t) \to 0 \) as \( t \to \infty \).

If \( \varphi(t) \equiv \text{const} \), then \( f \) is said to be normalized regularly varying (\( f \in \mathcal{NRV} \)).
Examples of (non-)SV functions

\[ f \text{ is regularly varying of index } \vartheta \text{ if and only if } \]

\[ f(t) = t^{\vartheta} L(t), \]

where \( L \in SV \).

- \( \prod_{i=1}^{n} (\ln_i t)^{\mu_i} \), where \( \ln_i t = \ln \ln_{i-1} t \) and \( \mu_i \in \mathbb{R} \) is \( SV \) function.

- \( 2 + \sin(\ln_2 t) \) and \( (\ln \Gamma(t))/t \) are \( SV \) functions.

- \( \frac{1}{t} \int_{a}^{t} \frac{1}{\ln s} \) \( ds \) is \( SV \) function.

- \( SV \) functions may exhibit “infinite oscillation” (i.e., \( \liminf_{t \to \infty} L(t) = 0 \), \( \limsup_{t \to \infty} L(t) = \infty \)), for example, \( \exp \left\{ (\ln t)^{1/3} \cos(\ln t)^{1/3} \right\} \).

- \( 2 + \sin t \), \( 2 + \sin(\ln t) \) are NOT \( SV \) functions.

- \( \exp t \) is NOT \( RV \) function.
Extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies “more slowly” than a power function.

$\mathcal{RV} \equiv$ generalized power laws;
(asymptotic scale invariant property)
$$f(\lambda t) = g(\lambda) f(t) \rightsquigarrow f(\lambda t) \sim g(\lambda) f(t) \text{ as } t \to \infty$$

Regularly varying functions have a “good behavior” with respect to integration and many other pleasant properties.

Regularly varying functions and related objects naturally occur in differential equations, and are very useful in the study of their qualitative properties.

Other selected properties

- If $L_1, L_2 \in SV$, then
  
  $L_1^\alpha \in SV$ (with $\alpha \in \mathbb{R}$), $L_1 L_2 \in SV$, $L_1 + L_2 \in SV$

  and

  $L_1 \circ L_2 \in SV$ (provided $L_2(t) \to \infty$).

- If $L \in SV$ and $\vartheta > 0$, then
  
  $t^\vartheta L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.  

Other selected properties

(Almost monotonicity) For a positive measurable function $L$ it holds: $L \in SV$ if and only if, for every $\vartheta > 0$, there exist a (regularly varying) nondecreasing function $F$ and a (regularly varying) nonincreasing function $G$ with

$$t^\vartheta L(t) \sim F(t), \quad t^{-\vartheta} L(t) \sim G(t) \quad \text{as } t \to \infty.$$  

(Asymptotic inversion) If $g \in RV(\vartheta)$ with $\vartheta > 0$, then there exists $g \in RV(1/\vartheta)$ with

$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as } t \to \infty.$$  

Here $g$ (an “asymptotic inverse” of $f$) is determined uniquely to within asymptotic equivalence. One version of $g$ is the generalized inverse $f^{-}(t) := \inf\{s \in [a, \infty) : f(s) > t\}$.  

Other selected properties (Karamata’s theorem)

(Karamata’s theorem; direct half)
(i) If \( L \in \mathcal{SV} \) and \( \gamma < -1 \), then
\[
\int_t^\infty s^\gamma L(s) \, ds \sim \frac{1}{-\gamma - 1} t^{\gamma + 1} L(t) \quad \text{as } t \to \infty.
\]
(ii) If \( L \in \mathcal{SV} \) and \( \gamma > -1 \), then
\[
\int_a^t s^\gamma L(s) \, ds \sim \frac{1}{\gamma + 1} t^{\gamma + 1} L(t) \quad \text{as } t \to \infty.
\]
(iii) The integral \( \int_a^\infty L(s)/s \, ds \) may or may not converge. The function
\( \tilde{L}(t) = \int_t^\infty L(s)/s \, ds \) or \( \tilde{L}(t) = \int_t^\infty L(s)/s \, ds \) is a new \( \mathcal{SV} \) function and such that \( L(t)/\tilde{L}(t) \to 0 \) as \( t \to \infty \).

(Karamata’s theorem; converse half)
Opposite directions in (i) and (ii).
Rapid variation

**Definition**

A measurable function $f : [a, \infty) \rightarrow (0, \infty)$ is called **rapidly varying of index** $\infty$, we write $f \in \mathcal{RPV}(\infty)$, if

$$
\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 
0 & \text{for } 0 < \lambda < 1, \\
\infty & \text{for } \lambda > 1,
\end{cases}
$$

and is called **rapidly varying of index** $-\infty$, we write $f \in \mathcal{RPV}(-\infty)$, if

$$
\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 
\infty & \text{for } 0 < \lambda < 1, \\
0 & \text{for } \lambda > 1.
\end{cases}
$$

The class of all rapidly varying solutions is denoted as $\mathcal{RPV}$.

While $\mathcal{RV}$ functions behaved like power functions (up to a factor which varies “more slowly”), $\mathcal{RPV}$ functions have a behavior close to that of exponential functions.

Regularly bounded functions, Zygmund class, ..., RV in other settings, ...
De Haan theory, class $\Pi$

**Definition**

A measurable function $f \in [a, \infty) \rightarrow \mathbb{R}$ is said to belong to the class $\Pi$ if there exists a function $w : (0, \infty) \rightarrow (0, \infty)$ such that for all $\lambda > 0$,

$$\lim_{t \rightarrow \infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln \lambda;$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function $w$ is called an *auxiliary function* for $f$.

The class $\Pi$ of functions $f$ is, after taking absolute values, a proper subclass of $SV$.

---

The de Haan theory is both a direct generalization of the Karamata theory and what is needed to fill certain gaps, or boundary cases, in Karamata’s main theorem.
Class \( \Pi \) – selected properties

- If \( f \in \Pi \), then for \( 0 < c < d < \infty \) the relation in the definition holds uniformly for \( \lambda \in [c, d] \).

- If \( f \in \Pi(L) \), then
  \[
  L(t) \sim f(t) - \frac{1}{t} \int_a^t f(s) \, ds
  \]
  as \( t \to \infty \).

- ...
De Haan theory, class $\Gamma$

**Definition**

A nondecreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the class $\Gamma$ if there exists a function $v : \mathbb{R} \to (0, \infty)$ such that for all $\lambda \in \mathbb{R}$

$$
\lim_{t \to \infty} \frac{f(t + \lambda v(t))}{f(t)} = e^\lambda;
$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$. The function $v$ is called an *auxiliary function* for $f$.

A function is said to belong to the class $\Gamma_-(v)$ if $1/f \in \Gamma(v)$.

It holds: $\Gamma_{\pm} \subset \mathcal{RPV}(\pm \infty)$

“Inversion” of $\Pi$, but there are also other way of understanding.
Class $\Gamma$ – selected properties

- If $f \in \Gamma$, then the relation in the definition holds uniformly on compact $\lambda$-sets.

- $f \in \Gamma$ if and only if

$$
\lim_{t \to \infty} \frac{f(t) \int_0^t \int_0^s f(\tau) \, d\tau \, ds}{\left( \int_0^t f(s) \, ds \right)^2} = 1.
$$

- If $f \in \Gamma(v)$, then

$$
v(t + \lambda v(t)) \sim v(t)
$$

as $t \to \infty$ locally uniformly in $\lambda \in \mathbb{R}$ (i.e., $v \in SN$; self-neglecting).

- ...
De Haan theory, Beurling slow variation

Definition
A measurable function $f : \mathbb{R} \to (0, \infty)$ is Beurling slowly varying if

$$\lim_{t \to \infty} \frac{f(t + \lambda f(t))}{f(t)} = 1 \quad \text{for all } \lambda \in \mathbb{R};$$

we write $f \in \mathcal{BSV}.$

Definition
If the relation holds locally uniformly in $\lambda$, then $f$ is called self-neglecting; we write $f \in \mathcal{SN}.$
Class $\mathcal{BSV} –$ selected properties

- If $f \in \mathcal{BSV}$ is continuous, then $f \in \mathcal{SN}$.
- $f \in \mathcal{SN}$ if and only if it has the representation
  \[ f(t) = \varphi(t) \int_0^t \psi(s) \, ds, \]
  where $\lim_{t \to \infty} \varphi(t) = 1$ and $\psi$ is continuous with $\lim_{t \to \infty} \psi(t) = 0$.
- If $f \in \mathcal{BSV}$ is continuous, then there exists $g \in C^1$ such that $f(t) \sim g(t)$ and $g'(t) \to 0$ as $t \to \infty$.
- ...
Regular variation and DEs

Theory of RV is “non-trivially” used; assumptions, statements, proofs.

Remark: In dozens of works on DEs, their authors work with “regular variation” without realizing it.

- **Linear and half-linear DEs**
  Geluk, Grimm, Hall, Jaroš, Kusano, Marić, Omey, Ř., Taddei, Tanigawa, Tomić

- **Quasilinear DEs and systems (equations of Emden-Fowler, Lane-Emden, Thomas-Fermi types)**
  Avakumović, Evtukhov, Jaroš, Kharkov, Kusano, Manojlović, Marić, Matucci, Radasin, Ř., Taliaferro, Tanigawa

- **PDEs**
  Cîrstea, Rădulescu, Zhang (uniqueness and asympt. behavior for solutions of nonlinear elliptic problems with boundary blow up); Niethammer, Pego (Ostwald ripening); Taliaferro (almost radial symmetry)
Regular variation and DEs

- **Stochastic (P)DEs and Volterra equations**
  Appleby, Debussche, Högele, Imkeller, Patterson

- **Difference equations**
  Ganelius, Geronimo, Manojlović, Kharkov, Kooman, Matucci, Ř., Smith, Van Assche
  (incl. orthogonal polynomials)

- **q-difference equations and dynamic equations on time scales**
  (dependence on the graininess)
  Ř., Vítovec

- ... Saari (*n*-body problem), Mijajlović, Segan (Friedmann equations), ...

Pavel Řehák (IM CAS)  RV and DEs  External meeting, Brno  19 / 40
Consider the quasilinear (or the generalized Emden-Fowler) equation
\[(\Phi_\alpha(y'))' = p(t)\Phi_\beta(y), \quad \Phi_\lambda(u) = |u|^\lambda \text{sgn } u,\] (E)
where \(\alpha, \beta > 0, \ p(t) > 0.\)

Basic classification of nonoscillatory solutions (with respect to the limit behavior as \(t \to \infty\) of a solution and its quasiderivative).

Problems of (non)existence in the classes.

A more precise description of asymptotic behavior of solutions (along with asymptotic formulae). In particular, we are interested in somehow difficult classes, e.g., the strongly increasing solutions (SIS), i.e.,
\[
\lim_{t \to \infty} y(t) = \infty = \lim_{t \to \infty} y'(t).
\]
\((\Phi_\alpha(y'))' = p(t)\Phi_\beta(y), \quad \Phi_\lambda(u) = |u|^{\lambda} \text{sgn } u, \quad (E)\)

- Kamo and Usami (2000, 2001) assumed \(p(t) \sim t^\sigma\) and \(\alpha > \beta > 0\). They established conditions (in terms of \(\alpha, \beta, \sigma\)) guaranteeing that SIS solutions \(y\) of \((E)\) have the form

\[y(t) \sim Kt^\gamma, \quad \text{where } K = K(\alpha, \beta, \sigma), \quad \gamma = \gamma(\alpha, \beta, \sigma).\]

The key role was played by the asymptotic equivalence theorem which says, roughly speaking:

*If the coefficients of two equations of the form \((E)\) are asymptotically equivalent, then their solutions which are “of the same class” are also asymptotically equivalent.*

- An extension of the classical results by Bellman, who assumed \(p(t) = t^\sigma\).
\[(\Phi_\alpha(y'))' = p(t)\Phi_\beta(y), \quad \Phi_\lambda(u) = |u|^\lambda \text{sgn } u, \quad p(t) \sim t^\sigma\]

We are interested in a generalization:

- A regularly varying coefficient
- A general coefficient in the differential term
- Regularly varying nonlinearities
- Coupled systems
- Second order systems of \(k\) equations (even-order scalar equations)
- First order systems of \(n\) equations (\(n\) may be even or odd)

- An asymptotic equivalence theorem cannot be used.
- The theory of RV in combination with some other tools finds the application.
- Much wider class of equations can be included.
Nonlinear system (of Emden-Fowler type)

\[
\begin{aligned}
    x'_1 &= a_1(t)F_1(x_2), \\
    x'_2 &= a_2(t)F_2(x_3), \\
    &\vdots \\
    x'_{n-1} &= a_{n-1}(t)F_{n-1}(x_n), \\
    x'_n &= a_n(t)F_n(x_1),
\end{aligned}
\]

(S)

\(n \in \mathbb{N}, n \geq 2.\)

- \(a_i\) are continuous, eventually of one sign, and

\[|a_i| \in \mathcal{RV}(\sigma_i), \quad \sigma_i \in \mathbb{R}, \quad i = 1, \ldots, n,\]

- \(F_i\) are continuous with \(uF_i(u) > 0\) for \(u \neq 0\), and

\[F_i(|\cdot|) \in \mathcal{RV}(\alpha_i), \text{ resp. } F_i(|\cdot|) \in \mathcal{RV}_0(\alpha_i), \quad \alpha_i \in (0, \infty), \quad i = 1, \ldots, n.\]

- A modification of our approach works also for some (more general) perturbed systems.
Special cases – $n$-th order two term nonlinear DE’s

- $x^{(n)} = p(t)\Phi_\beta(x)$

- or, more general,

$$D_{q_n}(\gamma_n)D_{q_{n-1}}(\gamma_{n-1}) \cdots D_{q_1}(\gamma_1)x(t) = p(t)\Phi_\beta(x),$$

where $D_{q_i}(\gamma_i)x(t) = \frac{d}{dt}(q_i(t)\Phi_{\gamma_i}(x))$

- The order can be even as well as odd.
Special cases – equations with a generalized Laplacian and/or an $\mathcal{RV}$ nonlinearity on the RHS

For example, the second order equation

$$(r(t)G(x'))' = p(t)F(x),$$

with

- a generalized Laplacian (may include the classical $p$-Laplacian operator or the curvature operator or the relativity operator or ...)
- a regularly varying nonlinearity on the right-hand side

Typical examples of nonlinearities (we do not require monotonicity):

- $F_i(u) = \Phi_\alpha(u) = |u|^\alpha \text{sgn } u$ (for second order equations or systems it may lead to the classical Laplacian operator).
- $F_i(u) = \Phi_\alpha(u)L(u)$, with $\alpha > 0$, where $L(u) \to c \in (0, \infty)$, or
  $L(u) = |\ln u|^{\gamma_1} |\ln |\ln u||^{\gamma_2}$.  
- $F_i(u) = u^\alpha (A + Bu^\beta)^\gamma$; a special choice yields
  $$\frac{u}{\sqrt{1 + u^2}} \text{ or } \frac{u}{\sqrt{1 - u^2}}.$$
Partial differential systems

System (S) includes also second order nonlinear systems of the form

\[
\begin{align*}
(A_1(t)\Phi_{\lambda_1}(y'_1))' &= B_1(t)G_1(y_2), \\
(A_2(t)\Phi_{\lambda_2}(y'_2))' &= B_2(t)G_2(y_3), \\
& \quad \vdots \\
(A_k(t)\Phi_{\lambda_k}(y'_k))' &= B_k(t)G_k(y_1),
\end{align*}
\]

which play important role in the study of positive radial solutions to the partial differential system

\[
\begin{align*}
\text{div}(\|\nabla u_1\|^\lambda_{1-1}\nabla u_1) &= \varphi_1(\|z\|)G_1(u_2), \\
\text{div}(\|\nabla u_2\|^\lambda_{2-1}\nabla u_2) &= \varphi_2(\|z\|)G_2(u_3), \\
& \quad \vdots \\
\text{div}(\|\nabla u_k\|^\lambda_{k-1}\nabla u_k) &= \varphi_k(\|z\|)G_k(u_1).
\end{align*}
\]

Systems of Lane-Emden type.
Extremal solutions

\[ x_i' = a_i(t)F_i(x_{i+1}), \quad i = 1, \ldots, n, \quad (S) \]

\( x_{n+1} \) means \( x_1 \).

- **Conditions on coefficients**: \( a_i \in \mathcal{RV}(\sigma_i), \ i = 1, \ldots, n. \)
- **Conditions on nonlinearities**: \( F_i \in \mathcal{RV}(\alpha_i) \);
  nonlinearities do not need to be monotone.
- **“Subhomogeneity”**: \( \alpha_1 \cdots \alpha_n < 1 \)
- Let us examine, for example, the class

\[ SIS = \{(x_1, \ldots, x_n) \in IS : \lim_{t \to \infty} x_i(t) = \infty, \ i = 1, \ldots, n\}; \]

the so-called strongly increasing solutions (or fast growing solutions).
The constants \( \nu_i \) and \( h_i \) and \( SV \) functions \( L_i \) depend on the coefficients and the nonlinearities; they can be explicitly computed.

**Theorem**

If \( \nu_i > 0, \ i = 1, \ldots, n \), then there exists

\[
(x_1, \ldots, x_n) \in SIS \cap (RV(\nu_1) \times \cdots \times RV(\nu_n))
\]

and (for every such a solution)

\[
x_i(t) \sim h_i t^{\nu_i} L_i(t) \quad \text{as } t \to \infty, \ i=1,\ldots,n.
\]  

(AF)

If \( \nu_i > 0 \) and, in addition, \( F_i = \Phi_{\alpha_i}, \ i = 1, \ldots, n \), then \( SIS \neq \emptyset \) and for EVERY \((x_1, \ldots, x_n) \in SIS\), one has

\[
(x_1, \ldots, x_n) \in RV(\nu_1) \times \cdots \times RV(\nu_n)
\]

and (AF) holds with \( L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1, \ L_{F_i}(t) = t^{-\alpha_i} F_i(t), \ i = 1, \ldots, n. \)

\[
\nu_i = \nu_i(\sigma_1, \ldots, \sigma_n, \alpha_1, \ldots, \alpha_n), \ h_i = h_i(\sigma_1, \ldots, \sigma_n, \alpha_1, \ldots, \alpha_n),
\]

\[
L_i = L_i(\sigma_1, \ldots, \sigma_n, \alpha_1, \ldots, \alpha_n, L_{a_1}, \ldots, L_{a_n}, L_{F_1}, \ldots, L_{F_n}).
\]
Sketch of the proof (the first part)

Properties of $\mathcal{RV}$ functions are frequently used ...

- The Schauder-Tychonoff fixed point theorem: We obtain a solution $(x_1, \ldots, x_n) \in SIS$ such that $c_i t^\nu_i L_i(t) \leq x_i(t) \leq d_i t^\nu_i L_i(t)$ for some constants $c_i, d_i, i = 1, \ldots, n$.

- $\lim \inf_{t \to \infty} x_i(\lambda t)/x_i(t), \lim \sup_{t \to \infty} x_i(\lambda t)/x_i(t) \in (0, \infty)$.

- The uniform convergence theorem and some other tools yield that $\lim_{t \to \infty} x_i(\lambda t)/x_i(t)$ exists; in fact,
  \[
  \limsup_{t \to \infty} x_i(\lambda t)/x_i(t) \leq \lambda^{\nu_i} \leq \liminf_{t \to \infty} x_i(\lambda t)/x_i(t).
  \]

Thus, $\mathcal{RV}$ follows.

- Playing with asymptotic relations and the Karamata theorem yield the asymptotic formula.
Sketch of the proof (the second part)

- $SIS \neq \emptyset$ follows from the previous part. We take an arbitrary $(x_1, \ldots, x_n) \in SIS$.

- Several auxiliary (quite) technical results, the generalized AM-GM inequality and some other estimates, and the properties of $\mathcal{RV}$ functions yield $c_i t^\nu L_i(t) \leq x_i(t) \leq d_i t^\nu L_i(t)$, $i = 1, \ldots, n$.

- $x_i \in \mathcal{RV}(\nu_i)$ and the asymptotic formula follow by similar arguments as in the first part.
We get back the Kamo-Usami result as a very special case.

In other ("less special") cases the results are new; even when \( a_i(t) \sim t^{\sigma_i} \) (for higher order) or for second order equations (with \( \mathcal{RV} \) coefficients).

For example, let \( p(t) = t^{\varrho} L_p(t) \) with \( L_p \in \mathcal{SV} \) and \( 0 < \beta < 1 \). If \( \varrho + 1 + \beta(n - 1) > 0 \), then the equation

\[
x^{(n)} = p(t) \Phi_\beta(x)
\]

possesses a solution \( x \) such that \( \lim_{t \to \infty} x^{(i)}(t) = \infty \), \( i = 0, \ldots, n - 1 \), and for any such a solution there hold

\[
x \in \mathcal{RV} \left( \frac{\varrho + n}{1 - \beta} \right) \quad \text{and} \quad x^{1-\beta}(t) \sim t^{\varrho + n} L_p(t) \prod_{j=1}^{n} \frac{1 - \beta}{\varrho + n - (1 - \beta)(j - 1)}.
\]

Regular variation of all these solutions is normalized.

We can treat also some equations with non-\( \mathcal{RV} \) coefficients via a suitable transformation.

Other types of solutions (decreasing, ...).

Other types of equations (perturbed equations, singular equations, superlinearity, ...).
Half-linear differential equations

\[(r(t)\Phi(y'))' = p(t)\Phi(y),\]  \hspace{1cm} (HL)

where \(r, p \in C([a, \infty)), \ r(t) > 0, \ \Phi(u) = |u|^{\alpha-1} \text{sgn} u \text{ with } \alpha > 1.\)

- The solution space is homogeneous but not additive.
- Bihari (1957, 1964), Elbert (1979), Mirzov (1976), ... dozens of authors
- If \(\alpha = 2\), then (HL) reduces to a linear Sturm-Liouville equation.
- Radially symmetric solutions of a certain PDE with \(p\)-Laplacian. If the dimension \(N = 1\), then (PDE) \(\equiv\) (HL).
- Special (bordeline) case of quasilinear (generalized Emden–Fowler) DEs.
- \(\alpha\)-degree functionals (variational principle, Hardy type inequalities, ...)

Pavel Řehák (IM CAS)  RV and DEs  External meeting, Brno
Solutions in the class $\Gamma$

$$(\Phi(y'))' = p(t)\Phi(y), \quad (HL)$$

where $p \in C([a, \infty)), p(t) > 0, \Phi(u) = |u|^{\alpha-1} \text{sgn } u$ with $\alpha > 1$.

Theorem

If

$$p^{-\frac{1}{\alpha}} \in BSV,$$

then all solutions of (HL) belongs to $\Gamma(v)$ or $\Gamma_-(v)$, where

$$v = \left(\frac{\alpha - 1}{p}\right)^{\frac{1}{\alpha}}.$$
Sketch of the proof
(here only for increasing solutions; decreasing solutions are more difficult)

- First assume $p \in C^1$. Then $(p^{-\frac{1}{\alpha}})'(t) \to 0$ as $t \to \infty$.
- Take $y$ such that $y > 0$, $y' > 0$. Set $w = p^{-\frac{1}{\beta}} \Phi(y'/y)$.
- $w$ satisfies the generalized Riccati equation

$$
\frac{w'}{p^{\frac{1}{\alpha}}(t)} = 1 - (\alpha - 1)w \left( \frac{p'(t)}{\alpha p^{\frac{\alpha+1}{\alpha}}(t)} + w^{\beta-1} \right)
$$

($\beta$ is the conjugate number to $\alpha$).
- $w$ satisfies $\lim_{t \to \infty} w(t) = (\alpha - 1)^{-\frac{1}{\beta}}$.
- $y$ satisfies

$$
\frac{y''(t)y(t)}{y'^2(t)} \sim 1
$$

- $y \in \Gamma \left( \left( \frac{\alpha-1}{p} \right)^{\frac{1}{\alpha}} \right)$. 
Sketch of the proof (continuation)

- We drop the assumption on differentiability of $p$.
- Since $p \in BSV$ there exists $\hat{p} \in C^1$ with $\hat{p}(t) \sim p(t)$ and $(\hat{p}^{-\frac{1}{\alpha}})'(t) \rightarrow 0$ as $t \rightarrow \infty$.
- For $\varepsilon \in (0, 1)$ we consider the auxiliary equations
  
  $(\Phi(u'))' = (1 + \varepsilon)\hat{p}(t)\Phi(u)$ and $(\Phi(v'))' = (1 - \varepsilon)\hat{p}(t)\Phi(v)$.

- For increasing solutions $u, v$ we show
  
  $$
  \left( \frac{u'(t)}{u(t)} \right)^{\alpha-1} \hat{p}^{-\frac{1}{\beta}}(t) \sim \left( \frac{1 + \varepsilon}{\alpha - 1} \right)^{\frac{1}{\beta}}, \quad \left( \frac{v'(t)}{v(t)} \right)^{\alpha-1} \hat{p}^{-\frac{1}{\beta}}(t) \sim \left( \frac{1 - \varepsilon}{\alpha - 1} \right)^{\frac{1}{\beta}}
  $$

- From the theorem on differential inequalities,
  
  $$
  \lim_{t \rightarrow \infty} \left( \frac{v'(t)}{v(t)} \right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) = (\alpha - 1)^{-\frac{1}{\beta}}.
  $$

- The rest of the proof is the same as in the previous part.
Remarks

- A part of the statement is a half-linear extension of known result. The other is new also in the linear case. (The Wronskian identity, reduction of order are not at disposal, ...)

- By-product: Condition for rapid variation of all solutions (open problem solved).

- By-product: There are solutions $y_1, y_2$ such that

$$y_i'(t) \sim \pm \left( \frac{p(t)}{\alpha - 1} \right)^{\frac{1}{\alpha}} y_i(t).$$

Half-linear extension of Hartman’s-Wintner’s result.

- Another generalization can be made.
Complete classification of solutions

\[ (r(t)\Phi(y'))' = p(t)\Phi(y), \]  

(KL)

Karamata theory, de Haan theory, generalized Riccati technique, reciprocity principle, asymptotic linearization, and other tools are used to obtain complete classification of solutions in the framework of regular variation.

— Relations among the classes; description of the structure of the solution space.

— Asymptotic formulae for ALL solutions.

— Interesting phenomena (that are not known from the linear case) are revealed

— Quite general setting that enables us to cover many important cases.

— \( \mathcal{S} \mathcal{V} \) components of the coefficients play an important role; more complex structure than with “asymptotically-power” coefficients.

— Sufficiency and necessity of the conditions.

— New also in the linear case.

An example of the result:

Pavel Řehák (IM CAS)

RV and DEs

External meeting, Brno
Consider the equation \((\Phi(y'))' = p(t)\Phi(y)\).

\(\mathcal{D}S\): eventually positive decreasing solutions;  
\(\mathcal{D}S_{A,B}\): \(\mathcal{D}S\) solutions with \(y \to A\), \(y' \to B\);  
\(S_{(N)SV}\): \((N)SV\)-solutions.

**Theorem**

Let \(p \in \mathcal{R}V(-\alpha)\). If \(L_p(t) \to 0\) as \(t \to \infty\), where \(L_p\) is \(SV\) component of \(p\), then \(\mathcal{D}S \subset \mathcal{N}SV\). For \(y \in \mathcal{D}S\), one has \(-y \in \Pi(-ty'(t))\). Moreover,

(i) If \(\int_{a}^{\infty} (sp(s))^{\frac{1}{\alpha-1}} \, ds = \infty\), then

\[
y(t) = \exp \left\{ - \int_{a}^{t} \left( \frac{sp(s)}{\alpha - 1} \right)^{\frac{1}{\alpha-1}} (1 + o(1)) \, ds \right\}
\]

as \(t \to \infty\), and \(S_{SV} = S_{NSV} = \mathcal{D}S = \mathcal{D}S_{0,0}\).

(ii) If \(\int_{a}^{\infty} (sp(s))^{\frac{1}{\alpha-1}} \, ds < \infty\), then

\[
y(t) = y(\infty) \exp \left\{ \int_{t}^{\infty} \left( \frac{sp(s)}{\alpha - 1} \right)^{\frac{1}{\alpha-1}} (1 + o(1)) \, ds \right\}
\]

as \(t \to \infty\), and \(S_{SV} = S_{NSV} = \mathcal{D}S = \mathcal{D}S_{B,0}, B > 0\).

In addition, \(|y(\infty) - y(t)| \in SV\), \(L_p^{\beta-1}(t)/(y(\infty) - y(t)) = o(1)\).
Sketch of the proof

- Take \( y \in \mathcal{DS} \).
- We show that \( ty'(t)/y(t) \to 0 \) as \( t \to \infty \), which implies \( y \in \mathcal{NSV} \).
- From \( -\Phi(y') \in \mathcal{RV}(1 - \alpha) \), we get \( -y(t) \in \Pi(-ty'(t)) \).
- Set \( h(t) = t^{\alpha - 1} \Phi(y'(t)) - (\alpha - 1) \int_a^t s^{\alpha - 2} \Phi(y'(s)) \, ds \).
- Then \( h \in \Pi(-(\alpha - 1)t^{\alpha - 1}\Phi(y'(t))) \) and \( h \in \Pi(th'(t)) \).
- From the uniqueness of the auxiliary function up to asymptotic equivalence and \( h'(t) = t^{\alpha - 1} p(t) \Phi(y(t)) \), we get
  \[
  \frac{y'(t)}{y(t)} = -(1 + o(1))\tilde{p}(t),
  \]
  where \( \tilde{p}(t) = \left( \frac{tp(t)}{\alpha - 1} \right)^{-1} \) as \( t \to \infty \).
- We distinguish the cases \( \int_0^\infty \tilde{p}(s) \, ds = \infty \) and \( \int_0^\infty \tilde{p}(s) \, ds < \infty \).
  Then we integrate the asymptotic relation to obtain formulas.
  Moreover, we find \( y(t) \to 0 \) resp. \( y(t) \to y(\infty) \in (0, \infty) \).
- Properties of \( \mathcal{RV} \) and \( \Pi \) functions yield the rest.


